V. "The Motion of Two Spheres in a Fluid." By W. M. Hicks, M.A., St. John's College, Cambridge. Communicated by Professor J. Clerk Maxwell, F.R.S., Professor of Experimental Physics in the University of Cambridge. Received May 16, 1878.

(Abstract.)

In the investigation the method of images is followed, and is based on the following lemma. The "image" of a source in an infinite fluid in presence of a sphere consists of a source at the inverse point of the former, and a line sink thence to the centre of the sphere. If the original source be μ at P, the inside source $=\frac{a}{\mathrm{OP}}\mu$, and the line density of the sink $=-\frac{\mu}{a}$, a being the radius of the sphere. For the analogous case of fluid within a sphere we, of course, require to have an equal source and sink somewhere within, else there must ensue motion across the boundary. In this case the analysis would give an infinite term for a single source, which, when the quantity of source and sink is zero, disappears. The image for a source would be as before a source $\frac{a}{\mathrm{OP}}\mu$ at the inverse point of P, and a line sink $-\frac{\mu}{a}$, thence to infinity. It is this last which produces the infinite term. From these lemmas are deduced the images of "doublets," by which are meant the singular points resulting from the coalescence of an equal source and sink whose magnitude varies inversely as the distance between them. It is easily seen that when the axis of a doublet passes through the centre of the sphere, its image is a single doublet within it, at the inverse point of the former of magnitude $-\left(\frac{a}{\mathrm{OP}}\right)^3\mu$, μ being the strength of the original doublet. A similar result holds for the image of a doublet within a sphere, whose axis passes through the centre of the sphere.

The image of a doublet whose axis is perpendicular to the line to the centre is not so simple. It consists as before of a doublet $\left(\frac{a}{\mathrm{OP}}\right)^3\mu$ at the inverse point, but besides this there stretches from the inverse point to the centre a negative line doublet of variable density, the line density at a distance, r, being $-\frac{\mu}{a}\frac{r}{\mathrm{OP}}$. The only difference between this and the case for an internal doublet is that the corresponding line doublet stretches to infinity.

No attempt is made to find the velocity potentials, but supposing them known, the kinetic energy of the motion for two spheres is

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expressed in terms of the magnitudes of the doublets also supposed known. The kinetic energy of the fluid motion is of the form

$$2T=A_1u_1^2+B_1(v_1^2+w_1^2)+A_2u_2^2+B_2(v_2^2+w_2^2)-2Lu_1u_2+2M(v_1v_2+w_1w_2),$$

where u_1, u_2 denote velocities along the line of centres, and v_1, w_1 . . . . perpendicular to it, and it is shown that

$$A_1=\frac{1}{2}m_1u_1^2\{1+3\Sigma_1^\infty\mu\},$$

where m_1 is the mass of fluid displaced by the sphere, and $\Sigma_1^\infty\mu$ denotes the sum of all the doublet-masses within (1) due to a unit motion of (1) along the line of centres. A similar formula holds for B_1, the $\Sigma\mu$ in this case denoting the doublet-masses within (1) due to a unit motion of (1) perpendicular to the line of centres. Also that

$$L=\frac{3}{2}m_2\Sigma_1^\infty(\mu')+\frac{3}{2}m_1\Sigma_1^\infty(\nu),$$

where $\Sigma_1^\infty(\mu')$ denotes summation for all the images inside (2) due to a unit motion of (1) along the line of centres, and $\Sigma_1^\infty(\nu)$ summation of all the images in (1) due to a corresponding motion of (2). An analogous expression holds for M. A_1, A_2, L, are completely determined in terms of the radii of the spheres and their distances, and an approximation is made to the values of B_1, B_2, and M. In certain cases the values for the coefficients take up simple forms; for instance, at the moment when one sphere is concentric with the bounding one $A_1=\frac{1}{2}m_1\frac{b^3+2a^3}{b^3-a^3}$, a result already obtained by Stokes.

The motion of the spheres along their line of centres is considered, and it is shown that whatever be their velocities, the effect of the fluid motion is such that relatively they appear to repel one another. More particularly too is considered the case of motion of a single sphere in an infinite fluid bounded by a plane. Amongst other things, it is shown that the ratio of the limiting velocity to the velocity when in contact with the plane $=\sqrt{\left\{\frac{2\rho+1\cdot6061707}{2\rho+1}\right\}}$, ρ being the density of the sphere.

Finally also is considered the effect of the fluid on vibratory motions and the mutual influence of two pendulums when they oscillate in their line of centres; data are given in the paper for the determination of the effect when they move in any manner. More particularly the effect on a sphere is considered when another has impressed on it a small harmonic vibration in the line of centres, and the force required to keep it at rest. In the first case the mean effect is an acceleration towards the vibrating sphere (A_1)

$$=-\frac{v^2}{A_2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right),$$

where v is the "velocity of mean square" of (A_1).

When the distance of the spheres is so large that we may neglect twelfth and higher inverse powers of the distance this attraction is

$$\frac{18v^2}{2\rho+1}\cdot\left(\frac{a}{r}\right)^7\frac{1}{a}\left\{\left(\frac{r^2}{r^2-b^2}\right)^4-\frac{3}{2\rho+1}\right\},$$

a, b, being the radii of the vibrating and free sphere respectively. From this it may be shown that if the density of the sphere is greater than the fluid it is attracted, whilst if less it will be attracted or repelled, according as its distance is less or greater than a certain critical distance, which is given by $r=\dfrac{b}{\sqrt{\left\{1-\sqrt[4]{\dfrac{2\rho+1}{3}}\right\}}}$. For instance, if $\rho=\cdot9$, this distance $=7\cdot648$, the radius of the free sphere. These formulæ are obtained on the supposition that the spheres are not so close that $\frac{dA_1}{dr}$ are too great to be neglected, for at contact $\frac{dA_1}{dr}$ are infinite. When the sphere is held fast, the mean force required to do so is $-v^2\frac{dA_1}{dr}$, which, when we neglect twelfth powers of $\frac{1}{r}$, $=\frac{v^2}{g}\frac{9a^3b^3r}{(r^2-b^2)^4}\times$ weight of fluid displaced by the vibrating sphere; for example, for two oxygen atoms at 0° C., at a distance four times their radius, the force is about 78×10^6 weight of fluid displaced by one atom, thus while the force decreases indefinitely, the effective force increases indefinitely.

For two spheres ($a=b=\frac{1}{4}$ inch $r=4a$) vibrating through a distance $\frac{1}{10}$th of an inch 256 times a second, in water, the force is equal to the weight of 12·8 milligrams.

VI. "Microscopical Researches in High Power Definition." By G. W. Royston-Pigott, M.A., M.D. Cantab., F.R.S. Received May 23, 1879.

(Abstract.)

In its general scope the paper is intended to deal with difficulties in microscopic research, usually found insuperable—such, for instance, as the invisibility of minute *closely packed* refracting spherules, existing in double rouleaux, or promiscuously aggregated, when their individual diameter varies between the 1–80,000th to the 1–200,000th of an inch.